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## SOLUTIONS OF EXERCISES.

## 62 and 63

FIND the radius of a spherical dome whose  $\frac{\text{volume}}{\text{surface}}$  is  $2000\frac{1}{8}$  and whose altitude is  $\frac{2}{3}$  of the radius.

SOLUTION.

If  $R$  be the radius of the sphere and  $H = \frac{2}{3}R$ , the height of the dome, we have for its surface,

$$S = 2\pi RH = \frac{4}{3}\pi R^2;$$

and for its volume,

$$V = \frac{2}{3}RS - \frac{1}{3}\pi(R-H)[R^2 - (R-H)^2] = \frac{2}{3}\pi R^3.$$

Hence in (62),

$$R = \sqrt[3]{\left(\frac{81000}{14\pi}\right)};$$

and in (63),

$$R = \sqrt{\left(\frac{1500}{\pi}\right)}. \quad [O. L. Mathiot.]$$

## 87

FIND the angle between the axes that  $x^2 + xy + y^2 = 0$  may be at right angles.

SOLUTION.

Solving with reference to  $y$ , we find that  $y = -(\frac{1}{2} \pm \frac{1}{2}i\sqrt{3})x$ . The condition that two lines shall be at right angles is, that

$$1 + (m + m') \cos \omega + mm' = 0.$$

These values for  $m$  and  $m'$ , when substituted in the equation of condition, give  $\cos \omega = 2$ , or the axes are imaginary. The angle between the axes must be  $\cos^{-1} 2$ . If the axes are real and orthogonal, the angle between the lines is  $\tan^{-1} \left( \frac{3i}{\sqrt{2}} \right)$ .

[Cooper D. Schmitt; R. H. Graves.]

## 88

An ellipse referred to equi-conjugate diameters inclined at  $\omega$ , has for equation  $x^2 + y^2 = c^2$ . Find the equation with reference to the same axes to the locus of the intersection point of orthogonal tangents.

SOLUTION.

The equation to the pair of tangents drawn from the point  $(x', y')$  is,

$$(x^2 + y^2 - c^2)(x'^2 + y'^2 - c^2) = (xx' + yy' - c^2)^2.$$

The condition that these lines should be at right angles gives

$$x'^2 + y'^2 + 2x'y' \cos \omega = 2c^2;$$

$\therefore$  the required equation is  $x^2 + y^2 + 2xy \cos \omega = 2c^2$ .

Or solve thus:—

The distance from the origin to the intersection point is equal to

$$\sqrt{a^2 + b^2} = \sqrt{(2c^2)};$$

$\therefore$  the required equation is  $x^2 + y^2 + 2xy \cos \omega = 2c^2$ .

[*R. H. Graves.*]

## 92

FROM the point of contact of two equal circles  $\alpha, \beta$  points  $A, B$  move on their circumferences with equal velocities in opposite directions. Find the motion of  $B$  relative to  $A$ .

SOLUTION.

Suppose the motion of the point  $A$  to be compensated by an equivalent motion of the plane of the two circles in the opposite direction. This motion of the plane is equal to that of the point  $B$ , and the motion of  $B$  is doubled. Hence the motion of  $B$  relative to  $A$  is the same as if  $A$  were to remain stationary, and  $B$  to describe a circle, whose diameter is twice that of either of the two equal circles.

The nature of the central force governing such motion may be determined as follows:—

The differential equation of the orbit pertaining to any attractive central force is

$$F = c^2 s^2 \left( \frac{d^2 s}{d\theta^2} + s \right), \quad (1)$$

in which  $c$  is some constant,  $s$  the reciprocal of the radius vector ( $s = 1/\rho$ ), and  $\theta$  the variable angle in polar co-ordinates, the pole being taken at the centre of force.

The polar equation of a circle, the pole being a point in the circumference, is

$$\rho = 2R \cos \theta,$$

from which

$$s = \frac{1}{2R} \sec \theta. \quad (2)$$

Differentiating (2),

$$\frac{ds}{d\theta} = \frac{1}{2R} \tan \theta \sec \theta,$$

and

$$\frac{d^2 s}{d\theta^2} = \frac{1}{2R} (\sec^3 \theta + \tan^2 \theta \sec \theta). \quad (3)$$

Substituting the values given in equations (2) and (3), in equation (1), and reducing,

$$F = 8R^2c^2s^5 = 8R^2c^2/\rho^5,$$

which shows that the force varies inversely as the fifth power of the distance. If the two particles,  $A$  and  $B$ , have equal masses, their motion, if governed by this law, will be that specified in the problem. [L. G. Weld.]

[Solved also by Professor Thornton.]

### 96

GIVEN on the ground a circular curve of known radius intersecting a given straight line at a given point, and inclined to it at that point at a given angle; it is required to determine the radius of a second circular arc which shall be tangent both to the given curve and to the given line at another given point.

[Calvin Whiteley.]

#### SOLUTION I.

Let  $R/P$  be the given curve,

$PQ$  the given line,

$JQ$  the required curve,

$C, C'$  the centres,

$R, R'$  the radii,

and  $PCM = 2A$  the given angle.

Put  $JCP = 2D, JC'Q = 2D'$ ,  
and draw the common tangent  
 $JT$ . In the triangle  $JPQ$

$$PQ = a,$$

$$PJQ = D + D',$$

$$JP = 2R \sin D,$$

$$JQP = D' = A - D,$$

$$JQ = 2R' \sin D',$$

$$JPQ = \pi - A - D';$$

since  $A = D + D'$ .

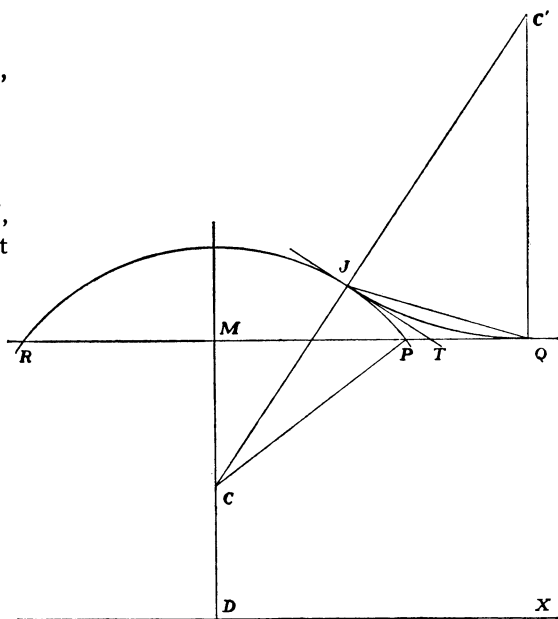
Accordingly

$$a : 2R \sin D : 2R' \sin D' = \sin A : \sin (A - D) : \sin (A + D'),$$

or

$$\frac{2R}{a} = \cot D - \cot A,$$

$$\frac{2R'}{a} = \cot D' + \cot A.$$



The first equation determines  $D$ ; then  $D' (= A - D)$  is known; and the second equation determines  $R'$ . [Frank Muller.]

SOLUTION II.

$C'$  is on a parabola,  $y^2 = 4dx$ ,

whose focus is  $C$ , and directrix  $DX$ , parallel to  $PQ$ , at a distance  $MD = R$ . We have at once  $2d = CD$ , or

$$\begin{aligned} d &= R \sin^2 A, \\ y &= \frac{1}{2}(a + b) \\ &= a + R \sin 2A, \\ R' &= x + d - R \\ &= x - R \cos^2 A \\ &= \frac{y^2}{4d} - R \cos^2 A \\ &= \frac{y^2 - R^2 \sin^2 2A}{4d}; \end{aligned}$$

whence

$$R' = \frac{a^2 + 2aR \sin 2A}{4R \sin^2 A}. \quad [S. M. Barton.]$$

SOLUTION III.

$$CC'^2 = MQ^2 + (CM + QC')^2;$$

$$\therefore (R + R')^2 = (a + R \sin 2A)^2 + (R' + R \cos 2A)^2,$$

whence by an easy reduction,

$$R' = \frac{a^2 + 2aR \sin 2A}{4R \sin^2 A}. \quad [J. E. Hendricks.]$$

SOLUTION IV.

With the same notations, let

$$QP = a, \quad QR = a + 2R \sin 2A = b, \quad CM = R \cos 2A = c;$$

then if  $TJ = TQ = t$ ,

$$t^2 = (a - t)(b - t),$$

$$\therefore t = \frac{ab}{a + b}.$$

From this result we compute

$$TCM = \tan^{-1} \frac{a + b - 2t}{c}, \quad TCJ = \tan^{-1} \frac{t}{R};$$

whence

$$2D' = TCM - TCJ,$$

and

$$R' = t \cot D'. \quad [Calvin Whiteley.]$$

## 98

Professor Hall calls attention to the fact that the reduction of

$$\begin{vmatrix} 1 & \cos p_1 & \sin p_1 \\ 1 & \cos p_2 & \sin p_2 \\ 1 & \cos p_3 & \sin p_3 \end{vmatrix}$$

required in the solution of the first part of this exercise, is to be found in Gauss's *Theoria Motus*, Art. 82.

Since the exercise was published, the proposer has found that the results are already given in Salmon's *Conic Sections*, Art. 231.\*

## 100

THE points  $O, O'$  defined by the equations in trilinears

$$aa : b\beta : c\gamma = c^2a^2 : a^2b^2 : b^2c^2,$$

$$aa : b\beta : c\gamma = a^2b^2 : b^2c^2 : c^2a^2,$$

are called the Brocard points. The angles  $OCA, OAB, OBC, O'BA, O'CB, O'AC$  are called the Brocard angles.

1. Show that the Brocard angles are all equal each to

$$\cot^{-1}[\cot A + \cot B + \cot C].$$

2. Find the equation to the Brocard line  $OO'$ .
3. Find the equation to the Brocard circle through  $O, O'$  and the centre of the circumscribed circle.
4. Given the base  $BC$  and the Brocard angle of a triangle, find the locus of the vertex.
5. Show that the bisectrices of the angles of  $ABC$  bisect the angles between the medians and the "symmedian lines"

$$\frac{a}{\alpha} = \frac{\beta}{b} = \frac{\gamma}{c}.$$

6. Show that the Brocard circle contains the symmedian point.

SOLUTION.

1. Let  $OCA = \omega$ . Then the equation of the line  $CO$  is

$$\frac{a}{\beta} = \frac{\sin(C - \omega)}{\sin \omega} = \sin C(\cot \omega - \cot C).$$

But by the given equations,

$$\frac{a}{\beta} = \frac{c^2}{ab} = \frac{\sin^2 C}{\sin A \sin B} = \sin C(\cot A + \cot B).$$

$$\therefore \cot \omega - \cot C = \cot A + \cot B,$$

or

$$\omega = \cot^{-1}(\cot A + \cot B + \cot C).$$

\* Prof. Barton also sent a correct solution of 98.

Similar reasoning gives the same value for each of the Brocard angles.

2.  $O$  is the intersection of the lines,

$$aba - c^2\beta = 0, \quad bc\beta - a^2\gamma = 0. \quad (1)$$

The equation of any line passing through  $O$  is

$$m_1aba + (m_2bc - m_1c^2)\beta - m_2a^2\gamma = 0.$$

So the equation of any line passing through  $O'$  is

$$m_3c^2a + (m_4a^2 - m_3ab)\beta - m_4bc\gamma = 0.$$

For the line  $OO'$  these equations must be identical. Hence by elimination and division we get

$$\frac{a}{a}(a^4 - b^2c^2) + \frac{\beta}{b}(b^4 - a^2c^2) + \frac{\gamma}{c}(c^4 - a^2b^2) = 0.$$

3. As all circles intersect in the circular points at infinity, a circle is expressed by the equation

$$a\beta\gamma + b\gamma a + c\alpha\beta + (aa + b\beta + c\gamma)(la + m\beta + n\gamma) = 0.$$

Substituting the values of  $a$  and  $\gamma$  given by equation (1), we get

$$lac^2 + mba^2 + ncb^2 + abc = 0 \quad (2)$$

as the condition that the circle passes through  $O$ . In like manner if it passes through  $O'$ ,

$$lab^2 + mbc^2 + nca^2 + abc = 0. \quad (3)$$

The centre of the circumscribed circle is the intersection of the two lines,

$$a \cos B = \beta \cos A, \quad \beta \cos C = \gamma \cos B.$$

Substituting these values of  $a$  and  $\beta$  in the general equation of the circle, and reducing by the formulæ,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

we get

$$l(a^3 - ab^2 - ac^2) + m(b^3 - bc^2 - ba^2) + n(c^3 - ca^2 - cb^2) - abc = 0. \quad (4)$$

Eliminating between (2), (3), and (4), we get

$$l = -\frac{bc}{a^2 + b^2 + c^2}, \quad m = -\frac{ac}{a^2 + b^2 + c^2}, \quad n = -\frac{ab}{a^2 + b^2 + c^2}.$$

This makes the equation of the required circle

$$abc(a^2 + \beta^2 + \gamma^2) - c^3a\beta - b^3\gamma a - a^3\beta\gamma = 0. \quad (5)$$

4. To express the locus of the point  $A$  in rectangular co-ordinates, make  $CB$  the axis of  $x$ , and  $C$  the origin. Then the point  $A$  is given by the equation,

$$\cot C = \frac{x}{y}, \quad \cot B = \frac{a-x}{y}, \quad \cot A = \frac{1 - \cot B \cot C}{\cot B + \cot C},$$

$$\cot \omega = \cot A + \cot B + \cot C,$$

as shown above. If  $\omega$  and  $a$  are given, the resultant of these equations is the equation to the circle,

$$ay \cot \omega = a^2 - ax + x^2 + y^2.$$

5. If in the equation  $a/a = \beta/b$  we interchange  $a$  and  $\beta$ , we get the equation of the median line  $\beta/a = a/b$ . But this interchange is equivalent to rotating the system about the bisectrix of  $C$ . Hence the bisectrices of the angles of  $ABC$  bisect the angles between the medians and the symmedians. The proposition as originally stated is incorrect.

6. In equation (5) put  $a, b, c$  for  $a, \beta, \gamma$ , and the equation is satisfied. Hence the Brocard circle (5) contains the symmedian point,

$$a:\beta:\gamma = a:b:c. \quad [C. B. Seymour.]$$

## 104

A SPIRAL of Archimedes having its pole in the circumference of a given circle passes through the centre and the point  $120^\circ$  of the circumference from the pole. Find its equation. [O. Root, Jr.]

SOLUTION.

Let  $\alpha$  be the angle made by the diameter through the pole with the initial axis. Then the curve

$$\rho = a\theta$$

passes through the centre, so that

$$R = a\alpha,$$

and also through the given point, so that

$$R(\sqrt{3} - 1) = (2n - \frac{1}{6})a\pi.$$

Hence

$$\alpha = \frac{R}{a}, \quad \frac{a}{R} = \frac{\sqrt{3} - 1}{(2n - \frac{1}{6})\pi}.$$

[R. H. Graves; Henry Heaton.]

## 105

IF the centre of the circumscribed circle of a triangle is on the circumference of the inscribed circle,

$$\cos \frac{1}{2}A \cdot \cos \frac{1}{2}A + \cos \frac{1}{2}B \cdot \cos \frac{1}{2}B + \cos \frac{1}{2}C \cdot \cos \frac{1}{2}C = 0.$$

If the same point is on the circumference of one of the escribed circles, a like relation holds. [W. M. Thornton.]

SOLUTION.

The equation to the inscribed circle is, in trilinears,

$$\cos \frac{1}{2}A \cdot a^{\frac{1}{2}} + \cos \frac{1}{2}B \cdot \beta^{\frac{1}{2}} + \cos \frac{1}{2}C \cdot \gamma^{\frac{1}{2}} = 0.$$

The co-ordinates of the centre of the circumscribed circle satisfy the equations,

$$\frac{a}{\cos A} = \frac{\beta}{\cos B} = \frac{\gamma}{\cos C}.$$

Hence the required relation follows.

Treat, in a similar manner the equations to the escribed circles, viz:—

$$\cos \frac{1}{2}A \cdot (-a)^{\frac{1}{2}} + \sin \frac{1}{2}B \cdot \beta^{\frac{1}{2}} + \sin \frac{1}{2}C \cdot \gamma^{\frac{1}{2}} = 0, \text{ etc.} \quad [R. H. Graves.]$$

106

If  $m$  be a positive integer,

$$\sin(m-1)\varphi - x \sin m\varphi + x^m \sin \varphi$$

will contain

$$1 - 2x \cos \varphi + x^2. \quad [A. Hall.]$$

SOLUTION.

The remainder, after a division of

$$R_n = x^n \sin(m-n-1)\varphi - x^{n+1} \sin(m-n)\varphi$$

by  $1 - 2x \cos \varphi + x^2$ , is by actual calculation found to be

$$R_{n+1} = x^{n+1} \sin(m-n-2)\varphi - x^{n+2} \sin(m-n-1)\varphi.$$

Accordingly, the division of

$$\sin(m-1)\varphi - x \sin m\varphi,$$

by  $1 - 2x \cos \varphi + x^2$ , will give the successive remainders,

$$R_1, R_2, R_3, \dots R_m,$$

where

$$R_m = x^m \sin(-\varphi).$$

This proves the proposition.

[*T. U. Taylor.*]

[Solved also by Professors Hall, Thornton, and Stone.]

## EXERCISES.

122

If  $P_1P_2P_3$  be a triangle inscribed in an ellipse, the co-ordinates of the point of concurrence of its altitudes are given by the relations,

$$2ax = (a^2 + b^2)(\cos p_1 + \cos p_2 + \cos p_3) - (a^2 - b^2) \cos(p_1 + p_2 + p_3),$$

$$2by = (a^2 + b^2)(\sin p_1 + \sin p_2 + \sin p_3) - (a^2 - b^2) \sin(p_1 + p_2 + p_3);$$

where  $p_1, p_2, p_3$  are the eccentric anomalies of the vertices. Prove these relations,